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U(N)-MONOPOLES ON KERR BLACK HOLE AND ITS ENTROPY *

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We describe U(N)-monopoles ($N > 1$) on Kerr black holes by the parameters of the moduli space of holomorphic vector U(N)-bundles over $\mathbb{S}^2$ with the help of the Grothendieck splitting theorem. For $N = 2, 3$ we obtain this description in an explicit form as well as the estimates for the corresponding monopole masses. This gives a possibility to adduce some reasonings in favour of existence of both a *fine structure* for Kerr black holes and the statistical ensemble tied with it which might generate the Kerr black hole entropy.

1. Introductory Remarks

The present paper is a natural continuation of our previous work of Ref.[1], so we shall not dwell upon the motivation of studying the topics being considered here so long as it has been done in Ref.[1]. It should be here only noted that one of the motivations of writing Ref.[1] was in the Kerr black hole case to realize the program performed in Refs.[2,3] for the Schwarzschild (SW) and Reissner-Nordström (RN) black holes, namely, to try finding the additional quantum numbers (nonclassical hair) characterizing Kerr black holes that might help in building a statistical ensemble necessary to generate the Kerr black hole entropy.

The mentioned program for SW and RN black holes consisted in that with the help of the classification of complex vector bundles over $\mathbb{S}^2$ and the Grothendieck splitting theorem a number of infinite series of U(N)-magnetic monopoles at $N \geq 1$ was constructed in an explicit form on the SW and RN black holes. Also the masses of the given monopoles were estimated to show that they might reside in black holes as quantum objects. This gave the possibility of applying to the problem of statistical substantiation of the SW and RN black hole entropy.[3]

The paper of Ref.[1] obtained some description of U(1)-monopoles on Kerr black holes. The present paper will be devoted to the extension of the constructions of Ref.[1] to the U(N)-monopoles ($N > 1$) on Kerr black holes along with an application to the problem of statistical substantiation of the Kerr black hole entropy. In the present paper, however, we shall use a gauge somewhat different from the gauge employed in Ref.[1] to avoid unnecessary complications.

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In the Kerr black hole case we use the ordinary set of the local Boyer-Lindquist coordinates t, r, ϑ, φ covering the standard topology $\mathbb{R}^2 \times \mathbb{S}^2$ of the $4D$ black hole spacetimes except for a set of the zero measure. At this the surface $t = const., r = const.$ is an oblate ellipsoid with topology $\mathbb{S}^2$ and the focal distance a while $0 \le \vartheta < \pi, 0 \le \varphi < 2\pi$. Under the circumstances we write down the Kerr metric in the form

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu \equiv (1 - 2Mr/\Sigma)dt^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\vartheta^2 -$$

$$[(r^2+a^2)^2 - \Delta a^2 \sin^2\vartheta]\frac{\sin^2\vartheta}{\Sigma}d\varphi^2 + \frac{4Mra\sin^2\vartheta}{\Sigma}dtd\varphi \qquad (1)$$

with $\Sigma = r^2 + a^2\cos^2\vartheta$, $\Delta = r^2 - 2Mr + a^2$, $a = J/M$, where J, M are, respectively, a black hole mass and an angular moment.

For inquiry we adduce the components of metric in the cotangent bundle of manifold $\mathbb{R}^2 \times \mathbb{S}^2$ with the metric (1) (in tangent bundle), so long as we shall need them in calculations below. These are

$$g^{tt} = \frac{1}{\Sigma\Delta}[(r^2+a^2)^2 - \Delta a^2\sin^2\vartheta], g^{rr} = -\frac{\Delta}{\Sigma}, g^{\vartheta\vartheta} = -\frac{1}{\Sigma},$$

$$g^{\varphi\varphi} = -\frac{1}{\Delta\sin^2\vartheta}(1 - 2Mr/\Sigma), g^{t\varphi} = g^{\varphi t} = \frac{2Mra}{\Sigma\Delta}\,. \qquad (2)$$

Besides we have $\delta = |\det(g_{\mu\nu})| = (\Sigma\sin\vartheta)^2$, $r_\pm = M \pm \sqrt{M^2 - a^2}$, so $r_+ \le r < \infty$, $0 \le \vartheta < \pi$, $0 \le \varphi < 2\pi$.

Throughout the paper we employ the system of units with $\hbar = c = G = 1$, unless explicitly stated. Finally, we shall denote $L_2(F)$ the set of the modulo square integrable complex functions on any manifold F furnished with an integration measure.

2. Description of U(N)-Monopoles

In order to obtain the infinite families of U(N)-monopoles for $N > 1$, we should use the Grothendieck splitting theorem[4,5] which asserts that any complex vector bundle over $\mathbb{S}^2$ (and, as a consequence, over $\mathbb{R}^2 \times \mathbb{S}^2$) of rank $N > 1$ [i. e., with the structural group U(N)] is a direct sum of N suitable complex line bundles over $\mathbb{S}^2$. The standard results of algebraic topology (see, e. g., Ref.[6]) say that U(N)-bundles over $\mathbb{S}^2$ are in one-to-one correspondence with elements of the fundamental group of U(N), π_1[U(N)]. On the other hand, in virtue of the famous Bott periodicity[7] $\pi_1[\mathrm{U(N)}] = \mathbb{Z}$ at $N \ge 1$ and, as a result, there exists the countable number of nontrivial complex vector bundles of any rank $N > 1$ over $\mathbb{R}^2 \times \mathbb{S}^2$. The sections of such bundles can be qualified as topologically inequivalent configurations (TICs) of N-dimensional (massless) complex scalar field. The above classification confronts some $n \in \mathbb{Z}$ with each U(N)-bundle over $\mathbb{R}^2 \times \mathbb{S}^2$ -topology. In what follows we shall call it the Chern number of the corresponding bundle. TIC with $n = 0$ can be called *untwisted* one while the rest of the TICs with $n \ne 0$ should be referrred to as *twisted*.

So far we tacitly implied that the U(N)-bundles were supposed to be differentiable. Really, they admit holomorphic structures and since each differentiable complex line bundle over $\mathbb{S}^2$ admits only one holomorphic structure (i. e., the holomorphic and differentiable classifications of complex line bundles over $\mathbb{S}^2$ coincide[4]) then the Grothendieck splitting theorem in fact gives a description of the moduli space $\mathfrak{M}_N$ of N-dimensional holomorphic complex vector bundles over $\mathbb{S}^2$. Namely, each N-dimensional holomorphic complex vector bundle over $\mathbb{S}^2$ is defined by the only N-plet of integers $(k_1, k_2, \ldots, k_N) \in \mathbb{Z}^N$, $k_1 \geq k_2 \geq \ldots \geq k_N$. Two of such N-plets (k_i) and (k_i') define the same differentiable N-dimensional bundle if and only if $\sum\limits_i k_i = \sum\limits_i k_i'$.

As was shown in Ref.[1], each complex line bundle (with the Chern number k_i, $i = 1, 2, ..., N$) over $\mathbb{R}^2 \times \mathbb{S}^2$ with the metric (1) has a complete set of sections in $L_2(\mathbb{R}^2 \times \mathbb{S}^2)$, so using the fact that all the U(N)-bundles over $\mathbb{R}^2 \times \mathbb{S}^2$ can be trivialized over the bundle chart of local coordinates $(t, r, \vartheta, \varphi)$ covering almost the whole manifold $\mathbb{R}^2 \times \mathbb{S}^2$, the mentioned set can be written on the given chart in the form

$$f^{a\omega_i}_{k_i l_i m_i} = \frac{1}{\sqrt{r^2+a^2}} e^{i\omega_i t} R^{a\omega_i}_{k_i l_i m_i}(r) Y_{k_i l_i m_i}(a\omega_i, \vartheta, \varphi)\,,$$
$$l_i = |k_i|, |k_i|+1, \ldots, |m_i| \leq l_i\,, \tag{3}$$

where some properties of both the *monopole oblate spheroidal harmonics* $Y_{k_i l_i m_i}(a\omega_i, \vartheta, \varphi)$ and the eigenvalues $\lambda_i = \lambda_{k_i l_i m_i}(a\omega_i)$ can be found in Ref.[1], but we shall not need them further. As to the functions $R^{a\omega_i}_{k_i l_i m_i}(r) = R$ then, in the gauge under discussion, they obey the equation

$$\frac{d}{dr}\Delta\frac{d}{dr}\left(\frac{R}{\sqrt{r^2+a^2}}\right) + \frac{(r^2+a^2)^2\omega_i^2 - 4Mm_i r a\omega_i + m_i^2 a^2}{\Delta}\frac{R}{\sqrt{r^2+a^2}} =$$
$$-(\lambda_i + k_i^2)\frac{R}{\sqrt{r^2+a^2}}\,, \tag{4}$$

with $l_i = |k_i|, |k_i|+1, \ldots, |m_i| \leq l_i$.

Now, in accordance with the Grothendieck splitting theorem, any section of N-dimensional complex bundle ξ_n over $\mathbb{R}^2 \times \mathbb{S}^2$ with the Chern number $n \in \mathbb{Z}$ can be represented by a N-plet $(\phi_1, \ldots, \phi_N)$ of complex scalar fields ϕ_i, where each ϕ_i is a section of a complex line bundle over $\mathbb{R}^2 \times \mathbb{S}^2$. According to the above, we can consider ϕ_i the section of complex line bundle with the Chern number $k_i \in \mathbb{Z}$, where the numbers k_i are subject to the conditions

$$k_1 \geq k_2 \geq \ldots \geq k_N\,,$$
$$k_1 + k_2 + \cdots + k_N = n\,. \tag{5}$$

As a consequence, we can require the N-plets $(f^{a\omega_1}_{k_1 l_1 m_1}, \ldots, f^{a\omega_N}_{k_N l_N m_N})$ to form the basis in $[L_2(\mathbb{R}^2 \times \mathbb{S}^2)]^N$ for the sections of ξ_n, $l_i = |k_i|, |k_i|+1, \ldots, |m_i| \leq l_i$, and

this will define the wave equation for a section $\phi = (\phi_1, \ldots, \phi_N)$ of ξ_n with respect to the metric (1)

$$\left[I_N \Box - \frac{1}{\Sigma^2 \sin^2\vartheta} \times \begin{pmatrix} 2ik_1\cos\vartheta(a\sin^2\vartheta\partial_t + \partial_\varphi) - k_1^2\cos^2\vartheta & 0 & \ldots & 0 \\ 0 & 2ik_2\cos\vartheta(a\sin^2\vartheta\partial_t + \partial_\varphi) - k_2^2\cos^2\vartheta & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 2ik_N\cos\vartheta(a\sin^2\vartheta\partial_t + \partial_\varphi) - k_N^2\cos^2\vartheta \end{pmatrix} \right] \times \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} = 0\,, \tag{6}$$

where I_N is the unit matrix $N \times N$, $\Box = (\delta)^{-1/2}\partial_\mu(g^{\mu\nu}(\delta)^{1/2}\partial_\nu)$ — the conventional wave operator conforming to metric (1).

The Eq. (6) will, in turn, correspond to the lagrangian

$$\mathcal{L} = \delta^{1/2} g^{\mu\nu}\overline{\mathcal{D}_\mu\phi}\mathcal{D}_\nu\phi\,, \tag{7}$$

with $\phi = (\phi_i)$ and a covariant derivative $\mathcal{D}_\mu = \partial_\mu - igA^a_\mu T_a$ on sections of the bundle ξ_n, while the overbar in (7) signifies hermitian conjugation and the matrices T_a will form a basis of the Lie algebra of U(N) in N-dimensional space (we, as is accepted in physics, consider the matrices T^a hermitian), $a = 1, \ldots, N^2$, g is a gauge coupling constant, i. e., we come to a theory describing the interaction of a N-dimensional twisted complex scalar field with the gravitational field described by metric (1). The coefficients A^a_μ will represent a connection in the given bundle ξ_n and will describe some nonabelian U(N)-monopole.

As can be seen, the Eq.(6) has the form $\mathcal{D}^\mu\mathcal{D}_\mu\phi = 0$, where $\mathcal{D}^\mu$ is a formal adjoint to $\mathcal{D}_\mu$ with regards to the scalar product induced by metric (1) in $[L_2(\mathbb{R}^2 \times \mathbb{S}^2\,)]^N$. That is, the operator $\mathcal{D}^\mu$ acts on the differential forms $a_\mu dx^\mu$ with coefficients in the bundle ξ_n in accordance with the rule

$$\mathcal{D}^\mu(a_\nu dx^\nu) = -\frac{1}{\sqrt{\delta}}\partial_\mu(g^{\mu\nu}\sqrt{\delta}a_\nu) + ig\overline{A_\mu}g^{\mu\nu}a_\nu \tag{8}$$

with $A_\mu = A^a_\mu T_a$.

As a result, the equation $\mathcal{D}^\mu\mathcal{D}_\mu\phi = 0$ takes the form

$$I_N\Box\phi - \frac{ig}{\sqrt{\delta}}\partial_\mu(g^{\mu\nu}\sqrt{\delta}A_\nu\phi) - (ig\overline{A_\mu}g^{\mu\nu}\partial_\nu + g^2 g^{\mu\nu}\overline{A_\mu}A_\nu)\phi = 0\,. \tag{9}$$

Comparing (6) with (9) gives a row of the (gauge) conditions:

$$A^a_r T_a = A^a_\vartheta T_a = 0\,, \tag{10}$$

$$g^{tt}A^a_t T_a + g^{t\varphi}A^a_\varphi T_a = \frac{a\cos\vartheta}{g\Sigma}\begin{pmatrix} k_1 & 0 & \ldots & 0 \\ 0 & k_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & k_N \end{pmatrix}\,, \tag{11}$$

$$g^{\varphi t}A_t^aT_a + g^{\varphi\varphi}A_\varphi^aT_a = \frac{\cos\vartheta}{g\Sigma\sin^2\vartheta}\begin{pmatrix} k_1 & 0 & \dots & 0 \\ 0 & k_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & k_N \end{pmatrix}, \tag{12}$$

This gives

$$A_t^aT_a = \frac{a\cos\vartheta}{g\Sigma}\begin{pmatrix} k_1 & 0 & \dots & 0 \\ 0 & k_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & k_N \end{pmatrix}, \tag{13}$$

$$A_\varphi^aT_a = -\frac{(r^2+a^2)\cos\vartheta}{g\Sigma}\begin{pmatrix} k_1 & 0 & \dots & 0 \\ 0 & k_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & k_N \end{pmatrix}. \tag{14}$$

Under the circumstances the connection in the bundle ξ_n is $A = A_\mu^aT_adx^\mu = A_t^a(r,\vartheta)T_adt + A_\varphi^a(r,\vartheta)T_ad\varphi$ which yields the curvature matrix $F = dA + A\wedge A$ for ξ_n-bundle in the form

$$F = F_{\mu\nu}^aT_adx^\mu\wedge dx^\nu = -\partial_r(A_t^aT_a)dt\wedge dr - \partial_\vartheta(A_t^aT_a)dt\wedge d\vartheta$$
$$+\partial_r(A_\varphi^aT_a)dr\wedge d\varphi + \partial_\vartheta(A_\varphi^aT_a)d\vartheta\wedge d\varphi + [A_t^aT_a, A_\varphi^bT_b]dt\wedge d\varphi\,, \tag{15}$$

because the exterior differential $d = \partial_tdt + \partial_rdr + \partial_\vartheta d\vartheta + \partial_\varphi d\varphi$ in coordinates t, r, ϑ, φ, while $[\cdot,\cdot]$ signifies the matrix commutator. Then, with taking into account Eqs. (13)–(14), we can see that the commutator in the right-hand side of (15) vanish and from here it follows that the first Chern class $c_1(\xi_n)$ of the bundle ξ_n can be chosen in the form

$$c_1(\xi_n) = \frac{g}{4\pi}\mathrm{Tr}(F)\,, \tag{16}$$

so that, when integrating $c_1(\xi_n)$ over any surface $t = const., r = const.$, we shall have with using (5) and (14)

$$\int\limits_{S^2} c_1(\xi_n) = \frac{g}{4\pi}\int\limits_{S^2}\mathrm{Tr}[\partial_\vartheta(A_\varphi^aT_a)]d\vartheta\wedge d\varphi = -\frac{n}{4\pi}\int\limits_{S^2}\Omega\sin\vartheta d\vartheta\wedge d\varphi =$$

$$-\frac{n}{2}\int\limits_0^\pi\Omega\sin\vartheta d\vartheta = n \tag{17}$$

with

$$\Omega = \frac{(r^2+a^2)(a^2\cos^2\vartheta - r^2)}{\Sigma^2}\,,$$

which is equivalent to the conventional Dirac charge quantization condition

$$qg = 4\pi n \tag{18}$$

with (nonabelian) magnetic charge

$$q = \int\limits_{S^2}\mathrm{Tr}(F)\,. \tag{19}$$

Introducing the Hodge star operator $*$ conforming metric (1) on 2-forms $F = F^a_{\mu\nu} T_a dx^\mu \wedge dx^\nu$ with the values in the Lie algebra of U(N) by the relation (see, e. g., Refs.[8])

$$(F^a_{\mu\nu} dx^\mu \wedge dx^\nu) \wedge (*F^a_{\alpha\beta} dx^\alpha \wedge dx^\beta) = (g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}) F^a_{\mu\nu} F^a_{\alpha\beta} \sqrt{\delta}\, dx^0 \wedge \cdots \wedge dx^3 \,, \tag{20}$$

written in local coordinates x^μ [there is no summation over a in (20)], in coordinates t, r, ϑ, φ we have for F of (15)

$$*F = *F^a_{\mu\nu} T_a dx^\mu \wedge dx^\nu =$$

$$(g^{t\varphi} g^{\vartheta\vartheta} \frac{\partial A_t}{\partial \vartheta} + g^{\vartheta\vartheta} g^{\varphi\varphi} \frac{\partial A_\varphi}{\partial \vartheta}) \sqrt{|\delta|}\, dt \wedge dr - (g^{\varphi t} g^{rr} \frac{\partial A_t}{\partial r} + g^{rr} g^{\varphi\varphi} \frac{\partial A_\varphi}{\partial r}) \sqrt{|\delta|}\, dt \wedge d\vartheta$$

$$+(g^{tt} g^{\vartheta\vartheta} \frac{\partial A_t}{\partial \vartheta} + g^{\vartheta\vartheta} g^{t\varphi} \frac{\partial A_\varphi}{\partial \vartheta}) \sqrt{|\delta|}\, dr \wedge d\varphi - (g^{tt} g^{rr} \frac{\partial A_t}{\partial r} + g^{rr} g^{t\varphi} \frac{\partial A_\varphi}{\partial r}) \sqrt{|\delta|}\, d\vartheta \wedge d\varphi \tag{21}$$

with $A_t = A^a_t T_a$ and $A_\varphi = A^a_\varphi T_a$ of (13)–(14). We can now consider the Yang-Mills equations

$$dF = F \wedge A - A \wedge F \,, \tag{22}$$

$$d * F = *F \wedge A - A \wedge *F \,. \tag{23}$$

It is clear that (22) is identically satisfied by the above A, F — this is just the Bianchi identity holding true for any connection.[8]

As for the Eq. (23), then, it is easy to check with the help of (13)–(14) and (21) that $*F \wedge A = A \wedge *F$. Under this situation, from (21) it follows that the condition $d * F = 0$ is equivalent to the equations

$$\frac{\partial}{\partial r}\left[\sqrt{|\delta|}\left(g^{rr} g^{\varphi t} \frac{\partial A_t}{\partial r} + g^{rr} g^{\varphi\varphi} \frac{\partial A_\varphi}{\partial r}\right)\right] + \frac{\partial}{\partial \vartheta}\left[\sqrt{|\delta|}\left(g^{t\varphi} g^{\vartheta\vartheta} \frac{\partial A_t}{\partial \vartheta} + g^{\vartheta\vartheta} g^{\varphi\varphi} \frac{\partial A_\varphi}{\partial \vartheta}\right)\right] = 0, \tag{24}$$

$$\frac{\partial}{\partial r}\left[\sqrt{|\delta|}\left(g^{tt} g^{rr} \frac{\partial A_t}{\partial r} + g^{rr} g^{t\varphi} \frac{\partial A_\varphi}{\partial r}\right)\right] + \frac{\partial}{\partial \vartheta}\left[\sqrt{|\delta|}\left(g^{tt} g^{\vartheta\vartheta} \frac{\partial A_t}{\partial \vartheta} + g^{\vartheta\vartheta} g^{t\varphi} \frac{\partial A_\varphi}{\partial \vartheta}\right)\right] = 0. \tag{25}$$

The direct evaluation with the aid of (13)–(14) shows that (24)–(25) are satisfied. As a consequence, the Eq. (23) is fulfilled.

One can notice, moreover, that

$$Q_e = \int\limits_{S^2} \mathrm{Tr}(*F) = -\int\limits_{S^2} g^{rr} \mathrm{Tr}\left(g^{tt} \frac{\partial A_t}{\partial r} + g^{t\varphi} \frac{\partial A_\varphi}{\partial r}\right) \sqrt{|\delta|} d\vartheta \wedge d\varphi = -\frac{4\pi a n r}{e} \int\limits_{-1}^{1} \frac{x\, dx}{\Sigma^2} = 0, \tag{26}$$

where $x = \cos\vartheta$. As a result, an external observer does not see any (internal) nonabelian electric charge Q_e of the Kerr black hole for any given N. Besides it should be emphasized that the total (internal) nonabelian magnetic charge Q_m of

black hole which should be considered as the one summed up over all the U(N)-monopoles for any given N remains equal to zero because

$$Q_m = \frac{4\pi}{g}\sum_{n\in\mathbb{Z}} n = 0\,, \tag{27}$$

so the external observer does not see any nonabelian magnetic charge of the Kerr black hole either though U(N)-monopoles are present on black hole in the sense described above.

To estimate the monopole masses we should use the T_{00}-component of the energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{4\pi}(-F^a_{\mu\alpha}F^a_{\nu\beta}g^{\alpha\beta} + \frac{1}{4}F^a_{\beta\gamma}F^a_{\alpha\delta}\,g^{\alpha\beta}g^{\gamma\delta}g_{\mu\nu})\,. \tag{28}$$

In our case

$$T_{00} = \frac{1}{4\pi}\{-g^{rr}(F^a_{tr})^2 - g^{\vartheta\vartheta}(F^a_{t\vartheta})^2 + \frac{1}{4}g_{tt}[g^{tt}g^{rr}(F^a_{tr})^2 + g^{tt}g^{\vartheta\vartheta}(F^a_{t\vartheta})^2 +$$
$$g^{rr}g^{\varphi\varphi}(F^a_{r\varphi})^2 + g^{\vartheta\vartheta}g^{\varphi\varphi}(F^a_{\vartheta\varphi})^2]\}\,, \tag{29}$$

where $F^a_{tr}T_a = -\partial_r(A^a_tT_a)$, $F^a_{t\vartheta}T_a = -\partial_\vartheta(A^a_tT_a)$, $F^a_{r\varphi}T_a = \partial_r(A^a_\varphi T_a)$, $F^a_{\vartheta\varphi}T_a = \partial_\vartheta(A^a_\varphi T_a)$.

Since we are in the asymptotically flat spacetime, we can calculate the sought masses according to

$$m_{\rm mon}(k_1,\ldots,k_N) = \int\limits_{t=const} T_{00}\sqrt{\gamma}\,dr\wedge d\vartheta\wedge d\varphi\,, \tag{30}$$

where

$$\sqrt{\gamma} = \sqrt{\det(\gamma_{ij})} = \sqrt{\Sigma/\Delta}\sin\vartheta\sqrt{(r^2+a^2)^2 - \Delta a^2\sin^2\vartheta} \tag{31}$$

for the metric $d\sigma^2 = \gamma_{ij}dx^i\otimes dx^j$ on the hypersurface $t = const$, while T_{00} is computed at the given U(N)-monopole. Under the circumstances it is not complicated to check that the leading term in asymptotic of $T_{00}\sqrt{\gamma}$ at $r\to\infty$ will be defined by the addend $g^{\vartheta\vartheta}g^{\varphi\varphi}(F^a_{\vartheta\varphi})^2$ of (29), so one should solve the equation

$$F^a_{\vartheta\varphi}T_a = \partial_\vartheta(A^a_\varphi T_a)\,, \tag{32}$$

with $A^a_\varphi T_a$ of (14). Let us concretize it for $N = 2, 3$.

3. Masses of U(2)- and U(3)-Monopoles

At $N = 2$ we can take $T_1 = I_2$, $T_a = \sigma_{a-1}$ at $a = 2, 3, 4$, where σ_{a-1} are the ordinary Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\sigma_2 = \begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix},\sigma_3 = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}\,. \tag{33}$$

Then the Eq. (32) gives $F^2_{\vartheta\varphi}=F^3_{\vartheta\varphi}=0$, while

$$F^1_{\vartheta\varphi}=\frac{1}{2}(k_1+k_2)f(r,\vartheta)\,,F^4_{\vartheta\varphi}=\frac{1}{2}(k_1-k_2)f(r,\vartheta) \tag{34}$$

with

$$f(r,\vartheta)=-\partial_\vartheta\left[\frac{(r^2+a^2)\cos\vartheta}{g\Sigma}\right]\,. \tag{35}$$

This yields at $r\to\infty$

$$T_{00}\sqrt{\gamma}\sim\frac{\sin\vartheta}{64\pi g^2r^2}[(k_1+k_2)^2+(k_1-k_2)^2]\,. \tag{36}$$

As a result, we can estimate (in usual units) according to (30)

$$m_{\rm mon}(k_1,k_2)\sim\left(\frac{\hbar^2c^2}{G}\right)\frac{(k_1+k_2)^2+(k_1-k_2)^2}{16g^2}\int\limits_{r_+}^{\infty}\frac{dr}{r^2}=\frac{(k_1+k_2)^2+(k_1-k_2)^2}{16g^2r_+}\left(\frac{\hbar^2c^2}{G}\right). \tag{37}$$

At $N=3$ we can take $T_1=I_3$, $T_a=\lambda_{a-1}$ at $a=2,\ldots,9$, where λ_{a-1} are the Gell-Mann matrices

$$\lambda_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\lambda_2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},\lambda_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},$$
$$\lambda_4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\lambda_5=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix},\lambda_6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},$$
$$\lambda_7=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\lambda_8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}. \tag{38}$$

From (32) this yields $F^2_{\vartheta\varphi}=F^3_{\vartheta\varphi}=F^5_{\vartheta\varphi}=F^6_{\vartheta\varphi}=F^7_{\vartheta\varphi}=F^8_{\vartheta\varphi}=0$, while

$$F^1_{\vartheta\varphi}=\frac{1}{3}(k_1+k_2+k_3)f(r,\vartheta),F^4_{\vartheta\varphi}=\frac{1}{2}(k_1-k_2)f(r,\vartheta),F^9_{\vartheta\varphi}=\frac{\sqrt{3}}{6}(k_1+k_2-2k_3)f(r,\vartheta) \tag{39}$$

with $f(r,\vartheta)$ of (35). This gives

$$m_{\rm mon}(k_1,k_2,k_3)\sim[(k_1+k_2+k_3)^2+\frac{9}{4}(k_1-k_2)^2+\frac{3}{4}(k_1+k_2-2k_3)^2]\frac{1}{36g^2r_+}\left(\frac{\hbar^2c^2}{G}\right). \tag{40}$$

It is clear that the case of arbitrary N can be treated analogously but we shall not dwell upon it here. One can only noticed that the important case is the one of U(4)-monopoles because 4-dimensional complex vector bundles could describe TICs of both spinors and vector charged fields, i. e. these TICs physically could arise due to interaction with U(4)-monopoles. But this task requires its separate consideration.

Under the circumstances, evaluating the corresponding Compton wavelength $\lambda_{\rm mon}(k_i) = \hbar/m_{\rm mon}(k_i)c$, we can see that at any $n \neq 0, N \geq 1$, $\lambda_{\rm mon}(k_i) \ll r_g$, where $r_g = r_+G/c^2$ is a gravitational radius of Kerr black hole, if $g^2/\hbar c \ll 1$. As a consequence, we come to the conclusion that under certain conditions U(N)-monopoles might reside in black holes as quantum objects.

So, we can see that the masses of U(N)-monopoles really depend on the parameters of the moduli space $\mathfrak{M}_N$ of holomorphic vector bundles over $\mathbb{S}^2$. Let us consider some possible issues for the $4D$ Kerr black hole physics from this fact.

4. Fine Structure of Kerr Black Hole for Generating Its Entropy

Among the unsolved questions of modern $4D$ black hole physics the so-called black hole information problem admittedly ranks high. Referring for more details, e. g., to Ref.[3] (and references quoted therein), it should be noted here that one aspect of the problem consists in that for an external observer any black hole looks like an object having in general only a finite number of parameters (classical hair — mass M, charge Q, angular momentum J) and it is, therefore, unclear how these parameters can encode all the information about quantum particles of matter (which has been collapsed to the black hole), particles that are being radiated à la Hawking. As a consequence, it is impossible to distinguish all the black hole (pure) states, so a black hole should, therefore, be described by a mixed state. In other words, the system (black hole) has an entropy S while the latter does not correspond to any statistical ensemble, so long as there is no infinite number of quantum (discrete) numbers connected with this system to build an appropriate statistical ensemble.

One can notice that recently the attempts have been undertaken to statistically substantiate the entropy for a range of black holes derived from string theory (see, e.g., Refs.[9] and cited therein). These black holes are, however, defined either in five dimensions or in four dimensions they carry a row of not yet observable quantum numbers, for example, the so-called axion charge. Therefore, such black holes cannot be used to describe real astrophysical objects and can only serve as some model examples. The real astrophysical objects having a claim on identifying with black holes seem to be described by the (SW, RN and Kerr) solutions derived from the standard Einstein gravity theory and we can call them *classical* black holes. It is clear that this is the most physically interesting set of black holes. But though for classical black holes also one can point out a number of attempts on statistical substantiation of their entropy, for example, within the framework of the so-called induced gravity (see, e.g., Refs.[10] and quoted therein), after all, these efforts have not yet led to any generally accepted statistical substantiation of the classical black hole entropy either. As a result, searching for new approaches to this problem for $4D$ classical black holes is well justified. In particular, in the above attempts the global nontrivial topological properties of black holes were practically ignored.

But the results of Refs.[2,3] for the SW and RN black holes as well as the ones of both Ref.[1] and the present paper for the Kerr black holes, however, show that

the natural candidates for additional quantum numbers (nonclassical hair) for classical black holes might be the quantum numbers parametrizing U(N)-monopoles on black holes, so these numbers could be identified with $\mathfrak{M}_N$. Really, as has been demonstrated recently in Refs.[1,11,12] black holes can radiate à la Hawking for any TICs, for instance, of complex scalar field with the Chern number $n \in \mathbb{Z} = \mathfrak{M}_1$ and this occurs independently of other field configurations. More exact analytical and numerical considerations[12] show that, for instance, in the SW black hole case, twisted TICs can give the marked additional contribution of order 17 % to the total luminosity (summed up over all the TICs). This tells us that there exists some *fine structure* in black hole physics which is conditioned by nontrivial topological properties of black holes and the given fine structure is able to markedly modify the black hole characteristics, so long as, for example, the words " Hawking radiation for complex scalar field " should be now understood as the radiation summed up over all the TICs of complex scalar field on black hole. This, in turn, leads to a marked increase of black hole luminosity.[1,12] In a sense, the black hole fine structure is quite analogous to the one of atomic spectra in atomic physics where its existence enables us to achieve an essentially better understanding of the whole structure of atoms.

Let us consider, therefore, more in detail in which way the above fine structure might help to Kerr black holes to form a statistical ensemble necessary to generate the Kerr black hole entropy.

As is known (see, e. g., Ref.[13]), the entropy S of Kerr black hole can be introduced from purely thermodynamical considerations and $S = \pi(r_+^2 + a^2)$, so when putting the internal energy of black hole $U = M$, we obtain the temperature of black hole $T = \frac{\partial U}{\partial S} = \frac{r_+ - r_-}{8\pi M r_+}$ through the standard thermodynamical relation. It is obvious that S corresponds to a formal partition function

$$Z = \exp\left[-\frac{M}{T} + \pi(r_+^2 + a^2)\right] . \tag{41}$$

The quantity Z is formal because we cannot point out any infinite statistical ensemble conforming to it, so that one could obtain Z by the usual Gibbs procedure, i. e., by averaging over this ensemble. The results of Ref.[1] show that Kerr black hole can radiate à la Hawking for any TIC of complex scalar field with the Chern number $n \in \mathfrak{M}_1 = \mathbb{Z}$. Such a radiation is practically defined by a couple $(g_{\mu\nu},\, n)$ with the black hole metric $g_{\mu\nu}$ of (1) and the Chern number n in the sense that these data are sufficient to describe the physical quantities (for instance, luminosity $L(n)$) characterizing the radiation process for TICs with the Chern number n.[1] On the other hand, as is known (see, e. g., Ref.[13]), the Hawking effect is being obtained when considering the system (black hole + matter field near it) semiclassically: the black hole is being described classically while the matter field is being quantized. All mentioned above suggests that the Hawking process occurs for the given pair $(g_{\mu\nu},\, n)$ when the black hole is in *a quantum state* which can be characterized by

the *semiclassical* energy

$$E_n \sim M - \frac{\sqrt{M^2-a^2}}{4Mr_+}(r_+^2+a^2) + \mathcal{E}(n) \tag{42}$$

with $\mathcal{E}(n) \sim m_{\rm mon}(n)Tr_+ \sim n^2T/4e^2$ with $m_{\rm mon}(n) \sim n^2/4e^2r_+$ of Ref.[1], $e = 4.8\cdot 10^{-10}\,{\rm cm}^{3/2}\cdot {\rm g}^{1/2}\cdot {\rm s}^{-1}$, so long as $\mathcal{E}(n)$ is a natural energy of the monopole with the Chern number n residing in Kerr black hole, since the additional contribution to the Hawking radiation is conditioned actually by the same monopole.[1] We call E_n semiclassical because the first two terms of (42) in usual units does not depend on $\hbar$ while the third addend does (see Sec. 3).

Under the circumstances there arises an infinite set of quantum states $(g_{\mu\nu},\, n)$ with the energy spectrum (42) for Kerr black hole. After this, the Gibbs average takes the form

$$Z \sim \sum_{n\in\mathbb{Z}} e^{-\frac{E_n}{T}} = \exp\left[-\frac{M}{T} + \pi(r_+^2+a^2)\right]\sum_{n\in\mathbb{Z}} e^{-\frac{n^2}{4e^2}} =$$
$$\exp\left[-\frac{M}{T} + \pi(r_+^2+a^2)\right]\vartheta_3(0,q) \tag{43}$$

with the Jacobi theta function $\vartheta_3(v,q)$ and $q = \exp\left(-\frac{1}{4e^2}\right)$. As a result, we obtain an inessential constant additive correction $S_1 = \ln\vartheta_3(0,q)$ independent of M and a to the Kerr black hole entropy $S = \pi(r_+^2+a^2)$ but now the latter is the result of averaging over an infinite ensemble which should be considered as inherent to Kerr black hole due to its nontrivial topological properties.

It is clear that one can also consider all the triplets $(g_{\mu\nu},\, k_1,\, k_2)$, where the pair $(k_1,\, k_2)$ parametrizes the moduli space of $U(2)$-monopoles $\mathfrak{M}_2$, so that the Gibbs average should be accomplished over $\mathfrak{M}_2$ which will again lead to some inessential additional correction to the entropy S due to dependence (37). Moreover, this scheme will obviosly hold true for U(N)-monopoles at any $N > 1$ if the Gibbs average is accomplished over the moduli space $\mathfrak{M}_N$.

5. Concluding Remarks

The results of both the present paper and Refs.[1,2,3,11,12] show that the $4D$ black hole physics can have a rich fine structure connected with the topology $\mathbb{R}^2\times\mathbb{S}^2$ underlying the $4D$ black hole spacetime manifolds. It seems to be quite probable that this fine structure is tied with the moduli spaces $\mathfrak{M}_N$ of N-dimensional holomorphic vector bundles over $\mathbb{S}^2$ and could manifest itself in solving the whole number of problems within the $4D$ black hole physics, so that one should seemingly thoroughly study the arising possibilities, in particular, also in the Kerr-Newman metric case as a natural charged generalization of Kerr metric.

On the other hand, the considerations of the present paper are actually of the general interest for all the metrics (solutions of the Einstein equations) which can naturally be realised on the topology $\mathbb{R}^2\times\mathbb{S}^2$. To this class of metrics one should, for

example, attribute the Kottler metric, Taub—NUT metric, the Vaidya metric (see, e. g., Ref.[14]). Especially, one should mark the class of Tomimatsu-Sato metrics[15] and their charged versions[16] which are natural extensions of Kerr and Kerr-Newman metrics.

We hope to realise such a study elsewhere.

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